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Construction of interface conditions for solving the compressible Euler equations by non-overlapping domain decomposition methods

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SUMMARY

In this work we examine the acceleration of the convergence of a non-overlapping additive Schwarz-type algorithm by modifying the transmission conditions applied to the subdomain interfaces. We have built generalized zero-order interface conditions using the Smith theory of diagonalizing polynomial matrices. The numerical experiments confirmed qualitatively the behaviour in accordance with the theory, but we could not reproduce identically the results obtained in the continuous case. The preliminary results are very encouraging since they lead to a very good convergence rate for certain Mach numbers. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: Euler equations; Schwarz algorithms; interface conditions; Fourier transform; Smith factorization

1. INTRODUCTION

We report on our recent efforts concerning the construction of non-overlapping additive Schwarz type algorithms for the solution of the system of Euler equations for compressible flows. We are specifically concerned with the construction of appropriate interface conditions that improve the convergence rate of the Schwarz algorithm. In Quarteroni and Stolcis [1], these transmission conditions are Dirichlet conditions for the characteristic variables corresponding to incoming waves. Such conditions can be qualified as classical interface conditions by opposition to more sophisticated formulations such as the optimized interface conditions studied in Reference [2] for an advection−diffusion equation. Here, we are interested in extending the principle of optimized interface conditions to the solution of the Euler equations. For this purpose, general type interface operators are introduced in the formulation of the additive Schwarz type algorithm. A convergence analysis is performed in the continuous case by

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considering the linearized Euler equations. An interface iteration is deduced from the formulation of the Schwarz algorithm in the Fourier space. In References [2–5], such a convergence analysis has been performed by applying a classical diagonalization method to the operator matrix involved in the problem. In this study, we apply the Smith factorization theory [5] in order to deduce a general form of the interface conditions. Then, the goal is to optimize the convergence rate with respect to certain parameters entering in the definition of these interface conditions. The analysis is limited to a two-subdomain decomposition in vertical strips.

2. DOMAIN DECOMPOSITION FOR THE EULER EQUATIONS

2.1. Mathematical model

The goal of the present study is to solve the time-dependent compressible Euler equations that can be written in conservative form as

$$\begin{gathered}\frac{\partial W}{\partial t} + \mathbf{\nabla}.\mathbf{F}(W) = 0 \\ W = (\rho, \rho\mathbf{U}, E)^{\mathrm{T}}, \quad \mathbf{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)^{\mathrm{T}}\end{gathered} \tag{1}$$

where $W = W(\mathbf{x}, t)$ is the vector of conservative variables; $\mathbf{x}$ and t, respectively, denote the spatial and temporal variables while $\mathbf{F}(W) = (F_x(W), F_y(W))^{\mathrm{T}}$ is the conservative flux whose components are given by

$$\begin{aligned}F_x(W) &= (\rho u, \rho u^2 + p, \rho uv, u(E + p))^{\mathrm{T}} \\ F_y(W) &= (\rho v, \rho uv, \rho v^2 + p, v(E + p))^{\mathrm{T}}\end{aligned}$$

In the above expressions, ρ is the density, $\mathbf{V} = (u, v)^{\mathrm{T}}$ is the velocity vector, E is the total energy per unit of volume and p is the pressure. The pressure is deduced from the other variables using the state equation for a perfect gas $p = (\gamma - 1)(E - \frac{1}{2}\rho\|\mathbf{V}\|^2)$, where γ is the ratio of specific heats ($\gamma = 1.4$ for the air). Under the hypothesis that the solution is regular one can also write a non-conservative (or quasi-linear) equivalent form of Equation (1):

$$\frac{\partial W}{\partial t} + A_x(W)\frac{\partial W}{\partial x} + A_y(W)\frac{\partial W}{\partial y} = 0 \tag{2}$$

where $A_x(W)$ and $A_w(W)$ are the Jacobian matrices of the flux vectors $F_x(W)$ and $F_y(W)$ (see Dolean [6] for more details). Suppose that we first proceed to an integration in time of (1) using a backward Euler implicit scheme involving a linearization of the flux functions. This operation results in the linearized system

$$\mathscr{L}(U) \equiv \frac{1}{\Delta t}U + A_x\frac{\partial U}{\partial x} + A_y\frac{\partial U}{\partial y} = f \tag{3}$$

where $U \equiv W^{n+1} - W^n$ where $W^{n+1} = W(x, (n+1)\Delta t)$, and A_x (respectively, A_y) is a shorthand for $A_x(W^n)$ (respectively, $A_y(W^n)$) and f is the right-hand side derived out of this linearization.

In the following we are interested in solving problem (3), associated to a suitable set of boundary conditions derived by a mathematical tool based on the Smith factorization [5] of the polynomial matrices, by a non-overlapping additive Schwarz type algorithm. An algorithm based on *classical transmission conditions* at subdomain interfaces that consist in Dirichlet conditions for the characteristic variables corresponding to incoming waves (a formulation already considered by Quarteroni and Stolcis [1]) has been studied in Dolean and Lanteri [7]. The main originality of this preliminary study is the analytical evaluation of the convergence rate of the Schwarz algorithm and the formulation in the discrete case of an interface system in terms of flux variables (the interface conditions being expressed in terms of upwind conservative normal fluxes computed using the approximate Riemann solver of Roe [8]). Time integration of the resulting semi-discrete equations is obtained using a linearized backward Euler implicit scheme. As a result, each pseudo-time step requires the solution of a sparse linear system for the flow variables, which is the discrete counterpart of (3).

2.2. Formulation of a Schwarz algorithm

For the simplicity of the formulation we consider a decomposition of the domain $[0,L]\times\mathbb{R}$ into vertical strips $([l_i,L_i]\times\mathbb{R})_{i\leqslant i\leqslant N}$ with or without overlaps. The linearized system (3) is solved by a Schwarz type algorithm. Let W_i^0 be the initial approximation of the solution in subdomain Ω_i. A general formulation of the additive Schwarz type algorithm for computing W_i^{k+1} from W_i^k (where k defines the iteration of the Schwarz algorithm) writes as

$$\begin{aligned} &A_x\frac{\partial W_i^{k+1}}{\partial x}+A_y\frac{\partial W_i^{k+1}}{\partial y}+BW_i^{k+1}=G \text{ in } (l_i,L_i)\times\mathbb{R}\\ &C_i^+(W_i^{k+1})=C_i^+(W_{i+1}^k) \text{ at } x=L_i\\ &C_i^-(W_i^{k+1})=C_i^-(W_{i-1}^k) \text{ at } x=l_i \end{aligned} \tag{4}$$

where the matrices $C_i^{\pm}$ have to be chosen so that the subproblems are well posed and the algorithm has a fast convergence rate. Natural (also qualified as classical) interface conditions resulting from the variational formulation of the initial and boundary value problem associated to system (1) are given by

$$C_i^+=A_x^+=T_x\Lambda_x^-T_x^{-1} \quad\text{and}\quad C_i^-=A_x^+=T_x\Lambda_x^+T_x^{-1} \tag{5}$$

where, in the present case, $A_x\equiv A_xn_x+A_yn_y$ since $n=(n_x,n_y)=(1,0)$ (as usual, n is the unitary external normal vector to the subdomain interfaces).

2.3. Convergence study of the Schwarz algorithm

In the following we proceed to the evaluation of the convergence rate of the Schwarz algorithm by means of a Smith factorization approach (see Reference [5]). This study is motivated by the need of a better understanding of the impact of the classical transmission conditions (5) on the convergence of the Schwarz algorithm (4). The result will confirm the one obtained by the eigenvectors approach that can be found in Reference [3]. The choice of Smith factorization as the mathematical tool for the convergence analysis is motivated by the fact that it yields a formulation which is intrinsic to the problem to be solved. In a second step, new interface conditions will be derived by generalizing the Smith form of the classical ones.

2.3.1. Smith factorization. The main result of the Smith factorization theory of polynomial matrices is recalled below.

Definition 1
Let $A(\lambda)$ be a $m \times m$ matrix with polynomials entries. There exist three matrices $E(\lambda)$, $D(\lambda)$ and $F(\lambda)$ with polynomial entries (where $E^{-1}(\lambda)$ and $F^{-1}(\lambda)$ have polynomial entries) where $\det(E(\lambda)) = \det(F(\lambda)) = 1$ and D is diagonal, such that

$$A(\lambda) = E(\lambda)D(\lambda)F(\lambda)$$

$D(\lambda)$ represents the Smith diagonal form of $A(\lambda)$; $E(\lambda)$ (respectively, $F(\lambda)$) is a permutation matrix that operates on the lines (respectively, the columns) of $A(\lambda)$. The entries of $D(\lambda)$ are given by $D_j(\lambda) = \phi_j(\lambda)/\phi_{j-1}(\lambda)$ where $\phi_j(\lambda)$ is the GCD of the determinants of all the $j \times j$ submatrices of $A(\lambda)$.

The first step consists in applying a Laplace transform in the x direction (the Laplace variable is denoted by λ) and a Fourier transform in the y direction (the Fourier variable is denoted by ξ) to system (3). The transformed system writes $A(\lambda,\xi)\hat{W} = \hat{f}$. The expression of the transformed matrix $A(\lambda,\xi)$ is given in Dolean [6]. $A(\lambda,\xi)$ being a polynomial matrix, it can be reduced to Smith form. In the present case, one obtains

$$D(\lambda,\xi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mathscr{G}(\lambda,\xi) & 0 \\ 0 & 0 & 0 & \mathscr{G}(\lambda,\xi)\mathscr{L}(\lambda,\xi) \end{pmatrix} \tag{6}$$

where

$$\begin{aligned} \mathscr{L}(\lambda,\xi) &= -(c^2-u^2)\lambda^2 + 2u(\beta + \mathrm{i}\xi v)\lambda + c^2\xi^2 + (\beta + \mathrm{i}\xi v)^2 \\ \mathscr{G}(\lambda,\xi) &= \lambda u + (\beta + \mathrm{i}\xi v) \end{aligned} \tag{7}$$

are the symbols of, respectively, a second-order elliptic operator and a transport operator.

2.4. Smith form of the Schwarz algorithm

Let $W = (w_1, w_2, w_3, w_4)^{\mathrm{T}}$ denote the vector of conservative variables and $\bar{W}_i := F(\partial_x, \xi)\bar{W}_i(x,\xi)$ the corresponding vector of Smith variables. The error vector (denoted here by $\bar{W}$) of the Schwarz algorithm satisfies

$$\begin{aligned} &D(\lambda,\xi)(\bar{W}_i^{k+1}) = 0 \\ &C_i^+(F^{-1}\bar{W}_i^{k+1}) = C_i^+(F^{-1}\bar{W}_{i+1}^k) \text{ at } x = L_i \\ &C_i^-(F^{-1}\bar{W}_i^{k+1}) = C_i^-(F^{-1}\bar{W}_{i-1}^k) \text{ at } x = l_i \end{aligned} \tag{8}$$

Because of the structure of the matrix D, it is sufficient to work with two Smith variables since we have $\bar{W}_{i,12}^{k+1} \equiv 0$, $\forall k \geqslant 0$. If we use the two subdomain case for the sake of the analysis and the hypothesis of a subsonic flow (the only case of interest for the convergence analysis of the Schwarz algorithm, see Reference [3] for more details), we have $\bar{W}_{1,3}^{k} \equiv 0$ and the algorithm can be written as

$$\begin{aligned}&\Omega_1 : \{b_{11}(\bar{W}_{1,4})^{k+1} = b_{11}(\bar{W}_{2,4})^k + b_{12}(\bar{W}_{2,3})^k \\ &\Omega_2 : \begin{cases} b_{21}(\bar{W}_{2,4})^{k+1} + b_{22}(\bar{W}_{2,3})^{k+1} = b_{21}(\bar{W}_{1,4})^k \\ b_{31}(\bar{W}_{2,4})^{k+1} + b_{32}(\bar{W}_{2,3})^{k+1} = b_{31}(\bar{W}_{1,4})^k \\ b_{41}(\bar{W}_{2,4})^{k+1} + b_{42}(\bar{W}_{2,3})^{k+1} = b_{41}(\bar{W}_{1,4})^k \end{cases}\end{aligned} \tag{9}$$

On the other hand, the local solutions are explicitly given by

$$\bar{W}_{1,4} = \alpha_1 e^{\lambda_{\mathscr{L}_1} x}, \quad \bar{W}_{2,4} = \alpha_2 e^{\lambda_{\mathscr{G}} x} + \alpha_3 e^{\lambda_{\mathscr{L}_2} x}, \quad \bar{W}_{2,3} = \alpha_4 e^{\lambda_{\mathscr{G}} x} \tag{10}$$

where $\lambda_{\mathscr{G}}$ and $\lambda_{\mathscr{L}_{1,2}}$ are the eigenvalues of the Fourier symbols $\Lambda_{\mathscr{G}}$ and $\Lambda_{\mathscr{L}_{1,2}}$ that factorize the operators $\mathscr{G}$ and $\mathscr{L}$, i.e. $\mathscr{G} = \partial_x - \Lambda_{\mathscr{G}}$ and $\mathscr{L} = (\partial_x - \Lambda_{\mathscr{L}_1})(\partial_x - \Lambda_{\mathscr{L}_2})$.

At that point, we can rewrite the interface iterations in terms of α_i, $i = 1,4$ and we get the expression of the convergence rate [6]

$$\rho(\xi, M_n, M_t) = \left| \frac{R(\xi) - a}{(R(\xi) + a)^2} \frac{R(\xi)(1 - 3M_n) - a(1 + M_n)}{1 + M_n} \right| \tag{11}$$

where $R(\xi) = \sqrt{a^2 + \xi^2(1 - M_n^2)}$, $a = 1/c\Delta t + i\xi M_t$ (M_n and M_t, respectively denote the Mach number normal and tangential to the interface with $M = \sqrt{M_n^2 + M_t^2}$, where M is the global Mach number) and c is the sound speed. We note that we get the same result as in the case of the eigenvectors approach [3].

3. GENERALIZED INTERFACE CONDITIONS

In the following, we derive new interface conditions by generalizing the Smith form of the Schwarz algorithm based on the classical interface conditions. Using the relation $(\bar{W}_{2,3})^{k+1} = b_{21}/b_{22}((\bar{W}_{1,4})^k - (\bar{W}_{2,4})^{k+1})$ we can rewrite the interface iterations (9) as

$$\begin{aligned}&\Omega_1 : \{b_{11}b_{22}(\bar{W}_{1,4})^{k+1} = (b_{11}b_{22} - b_{21}b_{12})(\bar{W}_{2,4})^k + b_{21}b_{12}(\bar{W}_{1,4})^{k-1} \\ &\Omega_2 : \begin{cases} (b_{31}b_{22} - b_{21}b_{32})(\bar{W}_{2,4})^{k+1} = (b_{32}b_{22} - b_{21}b_{32})(\bar{W}_{1,4})^k \\ (b_{41}b_{22} - b_{21}b_{42})(\bar{W}_{2,4})^{k+1} = (b_{41}b_{22} - b_{21}b_{42})(\bar{W}_{1,4})^k \end{cases}\end{aligned} \tag{12}$$

In order to obtain a general form of the iterations we introduce the operators $\mathscr{B}_i = p_i(\xi)\partial_x^2 + q_i\xi\partial_x + r_i\xi,\ i=1,4$ and we consider the following form of the Schwarz algorithm:

$$\begin{aligned}
&\Omega_1 : \begin{cases} \mathscr{L}(W_1^{k+1}) = 0 & \text{for } x<0 \\ \mathscr{B}_1(W_1^{k+1}) = (\mathscr{B}_1 + \mathscr{B}_2)(W_4^k) - \mathscr{B}_2(W_1^{k-1}) & \text{for } x=0 \end{cases} \\
&\Omega_2 : \begin{cases} \mathscr{L}(W_4^{k+1}) = 0 & \text{for } x>0 \\ \mathscr{B}_{3,4}(W_2^{k+1}) = (\mathscr{B}_{3,4}(W_1^k) & \text{for } x=0 \end{cases}
\end{aligned} \tag{13}$$

where $p_i(\xi)$, $q_i(\xi)$, $r_i(\xi)$ are polynomials in $\mathrm{i}\xi$.

From our point of view, the above two level iteration applied to the third-order partial differential equation $\mathscr{G}(\partial_x,\partial_y)\mathscr{L}(\partial_x,\partial_y)$ is a key ingredient in the good behaviour of the Schwarz algorithm based on the classical interface conditions. Then, our strategy consists in several steps (see Dolean [6] for more details). First, we derive a new form of the interface conditions in Smith variables by generalizing the expressions of $p_i(\xi)$, $q_i(\xi)$, $r_i(\xi)$. Then, we recover the physical interface conditions by requiring that they are easy to implement. While doing so, we have to check that the local problems are well posed and then estimate the convergence rate which depends on a few parameters. Finally, we can optimize the convergence rate with respect to these parameters

$$\begin{aligned}
&\min_{\sigma} \max_{\xi\in[0,\xi\,\max]} \rho(\xi,\sigma,M_\mathrm{n},M_\mathrm{t}) \text{ with} \\
&\rho(\xi,\sigma,M_\mathrm{n},M_\mathrm{t}) = \left| \frac{R(\xi)-a}{R(\xi)+a} \frac{R(\xi)(1-2M_\mathrm{n}-\sigma M_\mathrm{n}) - a(1+M_\mathrm{n})}{(1+M_\mathrm{n})(R(\xi)\sigma + a)} \right|
\end{aligned} \tag{14}$$

As a result of the optimization problem we obtain new conditions that lead to well-posed local problems for which the convergence rate ($\rho(\xi)$) becomes null at two wavenumbers (instead of only one at $\xi=0$ for the Schwarz algorithm based on the classical interface conditions).

4. NUMERICAL RESULTS

4.1. Space and time discretization methods

The spatial discretization method adopted here combines the following elements (see Dolean and Lanteri [7] for more details): (1) a finite volume formulation on triangular meshes together with upwind schemes for the discretization of the convective fluxes; (2) an extension to second-order accuracy that relies on the monotonic upstream schemes for conservation laws (MUSCL) to unstructured triangular meshes by Fezoui and Stoufflet [4]. Time integration of the resulting semi-discrete equations is obtained using a linearized backward Euler implicit scheme [4]. As a result, each pseudo-time-step requires the solution of a sparse linear system for the flow variables. In this study, a non-overlapping domain decomposition algorithm implemented on a parallel architecture of Pentium Pro is used for advancing the solution at each implicit time step.

Table I. Non-overlapping additive Schwarz type algorithm. Classical interface conditions versus generalized interface conditions.

M_n	OPT0	OPT1	M_∞	OPT0	OPT1
0.1 and $M_t = 0.0$	20	20	0.3 and $M_t = 0.0$	24	19
0.6 and $M_t = 0.0$	27	17	0.1 and $M_t = 0.1$	24	21
0.3 and $M_t = 0.2$	24	28	0.6 and $M_t = 0.4$	32	18
0.6 and $M_t = 0.7$	25	21	0.8 and $M_t = 0.5$	42	21

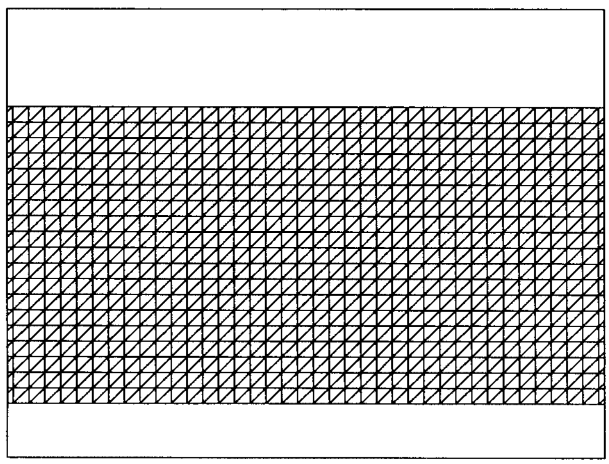

Figure 1. Structured triangular mesh of a rectangular domain.

4.2. Numerical results

We present here a set of preliminary results of numerical experiments that are concerned with the evaluation of the influence of the interface conditions on the convergence of the non-overlapping additive Schwarz type algorithm of form (4). The computational domain is given by the rectangle $[0,2] \times [0,1]$. The numerical investigation is limited to the resolution of the linear system resulting from the first implicit time step using a Courant number $\mathrm{CFL} = 100$. A slipping condition ($\mathbf{V}.\mathbf{n} = 0$) is applied on the lower ($y = 0$) and upper ($y = 1$) walls; an inflow (respectively, outflow) condition is applied on the left $x = 0$ (respectively, right $x = 10$) boundary. Table I summarizes the number of Schwarz iterations required to reduce the initial linear residual by a factor 10^{-10} for different values of the reference Mach number. The underlying triangular mesh is a regular one deduced from a finite difference grid containing 4000 nodes (200×20) (see Figure 1). In this table, OPT0 stands for the classical interface conditions while OPT1 corresponds to the algorithm based on the generalized interface conditions.

4.3. Conclusions

In this work we were interested in the acceleration of the convergence of a non-overlapping additive Schwarz type algorithm by modifying the transmission conditions applied to the subdomain interfaces. We built generalized zero order interface conditions using Smith theory of diagonalizing polynomial matrices. The numerical experiments confirmed at least qualitatively the behaviour in accordance with the theory even if from the discrete point of view we could not reproduce identically the results obtained in the continuous case. The preliminary results are very encouraging as they lead to a very good convergence rate for certain Mach numbers.

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